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Dynamics of route choice and signal control in capacitated networks

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Abstract

This paper considers the stability of the dynamical system which arises when a responsive control system is utilised in a signal-controlled urban road network. In this case current traffic flows change current green-times (according to the responsive control policy) and current green-times change current delays and hence drivers' route choices and current flows. Simple networks only are considered; starting with a small symmetrical network with two routes and a traffic signal which follows the equisaturation control policy. The symmetrical equilibrium, with equal flows on both routes, is (under reasonable conditions) unstable. The paper then shows that, within the simple network considered, bottlenecks may be added which makes the symmetrical equilibrium stable for certain steady Origin-Destination loads. Finally the paper considers the stability of a similar asymmetrical network when a special responsive policy is used. Under natural conditions the network is shown to be stable under this policy for all feasible Origin-Destination loads. The stability proof given for this network is designed to generalise so as to apply to a general signal-controlled network under suitable conditions; however such a general stability proof is not included in this paper.

Keywords: Route choice, Signal control, Dynamics of route choice and signal control.

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1. Introduction

1.1 Outline of the paper

This paper considers simple models of routeing changes as drivers seek better routes and signal control changes as an adaptive control system responds to traffic flows.

We start by considering the responsive equisaturation control policy on a simple three-link network. In this network; for each origin-destination (OD) load the symmetrical equilibrium, with equal flows on both routes, is unstable. The paper shows that bottlenecks may be added to this network which make the symmetrical equilibrium stable for most Origin destination loads. The added bottlenecks reduce the unpredictability of the network very substantially although a little of the rather complicated behaviour familiar in complex systems is present.

Then the paper considers the stability of a similar network with the same delay formulae, when a new traffic control policy very similar to the P_0 policy (see Smith 1979a, b) is utilised. It is shown that in this case the natural adjustment of routeing and green-times becomes stable, and there is then of course no need to improve that stability with added bottlenecks. The proof of stability given here appears to generalise so that it applies to a general network. A related more general stability result, but in a very different context, is given in Smith and Mounce (2011).

1.2 The P_0 control policy

The P_0 policy on a junction with just two approaches seeks green times which equalise the following two values:

the saturation flow on approach 1 × average delay felt on approach 1 = s_1d_1

and

the saturation flow on approach 2 × average delay felt on approach 2 = s_2d_2.

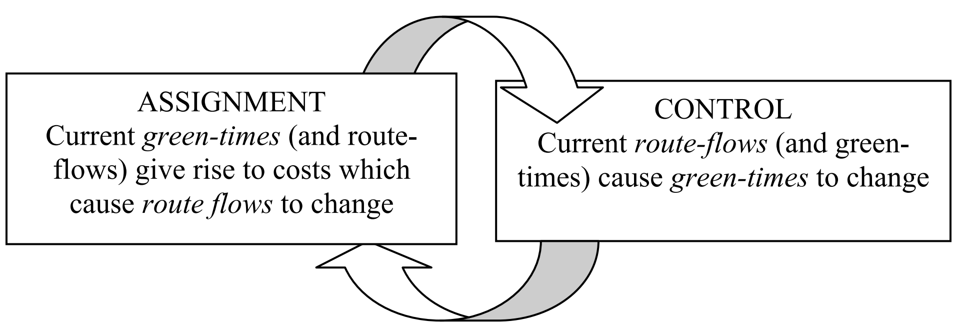

Figure 1. A simple representation of a dynamical system arising when a responsive control system is utilised. This loop in which current route-flows change current green-times (according to some responsive control policy) and current green-times change current delays and hence current flows (according to drivers' route-choices) may be regarded as being traversed indefinitely.

When faced with two approaches where $s_1d_1 < s_2d_2$ the policy transfers some of the green-time from approach 1 to approach 2. (This will normally tend to increase s_1d_1 and reduce s_2d_2.) The green-time transfer will continue until either $s_1d_1 = s_2d_2$ or the green-time awarded to approach 1 is a minimum. This statement of the policy generalises easily so as to apply to a more complicated junction with different stages during which different sets of links are shown green.

The main difference between the P_0 policy and standard policies is that with P_0 the two measurements of "disbenefit" being equalised above (s_1d_1 and s_2d_2) do not explicitly involve the flow volumes on the two approaches. The policy is less driven by the flows themselves than standard policies, such as equisaturation or delay-minimisation.

The central property of the P_0 policy is as follows. Suppose that some responsive policy P is applied locally at each junction of an arbitrary capacitated network N. Suppose that there is a given Origin-Destination matrix M. Let K be the greatest multiplier of the given Origin-Destination matrix M which has a feasible equilibrium consistent with the responsive control policy P. Then K depends on the network N, the OD matrix M and the responsive policy P; so K = K(N, M, P) and it is natural to seek a policy P which maximise K(N, M, P). It turns out that K(N, M, P) is, under reasonable conditions, maximised by choosing the policy P to be the P_0 policy. See Smith (1979a, b, 1987). The policy achieves this capacity-maximisation effect by seeking to ensure that approaches with high saturation flows at a junction have lower delays, encouraging drivers to use scarce junction capacity resource economically. The policy "prices" the approaches suitably using delays instead of prices. *The conditions required to prove the above result preclude blocking back.*

1.3 Context

Both traffic control and route choice have been studied very widely indeed. A very highly selective list of references is given below. These papers address one or more of the following five topics:

1 Equilibrium route choice modelling;
2 Non-equilibrium route choice modelling;
3 Signal control modelling;
4 Equilibrium route choice and signal control modelling; and
5 Non-equilibrium route choice and signal control modelling.

Below, all references have been placed approximately chronologically within the most relevant section: section 1.2. x below deals with topic x above for x = 1, 2, 3, 4, 5.

Many of these papers discuss models which are not explicitly motivated by the concept of random utility; and random utility is rarely if ever mentioned. Those route choice models which are stochastic are random utility models; and almost all the models discussed are utility maximising models. The combined dynamics of signal timings and route choices have been very rarely studied.

The main aspect missing from almost all these studies below (and also missing from this study) is blocking back; which happens if a junction is blocked by queues arising from downstream bottlenecks.

Cascetta (2009) addresses several of the issues also studied in the papers below. Bell (1992) conceives of integrating traffic signal control with other aspects such as road pricing and information provision.

1.3.1 Equilibrium route choice modelling

Transportation planning depends critically on good models estimating how OD flows are likely to spread over the links of a network; and the simplest such models are route choice models. Wardrop (1952) is credited with the first clear statement that the total OD flow is likely to distribute itself over the links of a network so that:

for each OD pair more costly routes are not used.

The central paper on this equilibrium subject is due to Beckmann et al. (1956) – this paper allowed for elastic demand and capacity constraints. Smith (1979b, 1984b,c, 1987), Dafermos (1980), Fisk (1980, 1984), Aashtiaani and Magnanti (1983), Sheffi (1985) and Cantarella (1997), among many others; have extended this theory conceptually and algorithmically. Kupiszewska and Van Vliet (1999) make a strong computational case for utilising route-flows (rather than link flows) in traffic assignment programs.

Recent work by Bar Gera (2002, 2010) and Bar Gera and Boyce (2003, 2006) using approach proportions merits emphasis. This has transformed both the accuracy and the convergence speed (when high accuracy is required) of equilibration algorithms. Gentile (2009) and Gentile and Noekel (2009) have implemented equilibration algorithms using splitting rates (or leaving proportions) at nodes rather than approach proportions.

Dynamic *equilibrium* models have been considered by very many including, for example, de Palma et al. (1983), Carey (1987), Cascetta (1991), Ran and Boyce (1996) and Friesz and Mookherjee (2006). A recent exposition of the dynamic equilibrium problem is provided by Friesz and Bernstein (2007); the references therein provide further information on this important topic.

1.3.2 Non-equilibrium route choice modelling

Non-equilibrium route choice models seek to represent the non-equilibrium route choice behaviour of travellers or drivers. There is an increasing number of these including models outlined by Smith (1984a), Cantarella and Cascetta (1995), Watling (1996, 1999), Mahmassami and Liu (1999), Watling and Hazelton (2003), Bellei et al. (2005), Hamdar et al. (2008), He et al. (2010), Bie and Lo (2010) and Smith and Mounce (2011).

1.3.3 Signal control modelling

Webster (1958) was one of the first to seek to model signal timings and their effect on traffic flow at a single junction. Robertson (1969) gives a model of a whole network (TRANSYT) allowing whole network optimisation of traffic signals (for know OD inputs and known routes). The subject is a very large one; Wood (1993) provides a review of certain urban traffic control systems.

1.3.4 Equilibrium route choice and signal control modelling

Allsop (1974) pointed out the importance of allowing for route choices when considering the impacts of signal control changes. Gartner (1976) considers area traffic control and network equilibrium and Gartner (1983) specifies a traffic control policy which responds to varying demand. Allsop and Charlesworth (1977) gave an example where different equilibrium routeings arise from the same control policy, and Dickson (1981) showed that optimising signals for fixed flows does not give optimum timings when route choices are variable.

Smith (1979a,b) specified a local traffic control policy which under certain conditions automatically maximises the overall travel capacity of a signal controlled network, allowing route choices to vary. Bentley and Lambe (1980) showed how green times and traffic flows may be combined within a single assignment model. There are strong connections between these two papers.

Smith (1987, 2010), Van Vuren and Van Vliet (1992), Smith and van Vuren (1993), Yang and Yagar (1995) and Yang (1996) have considered in detail the interaction between signal control and routeing. Meneguzzer (1996, 1997) reports computational experiments with combined traffic assignment and control models and provides a review of models linking signal control and route choice. Taale and van Zuylen (2001) provide an interesting discussion of their own work and the work of others in combining signal control and route choice.

Smith (2009) gives a two direction method of calculating variable or fixed demand equilibria consistent with the P_0 policy. Mounce (2009) has shown that a time-varying equilibrium exists with responsive control (under conditions which prohibit blocking back).

Bilevel programming has been applied to optimising urban traffic signal settings and prices subject to equilibrium routeing. See for example Marcotte (1983), Sheffi and Powell (1983), Clegg et al. (2001), Cipriano and Fusco (2004), Cascetta et al. (2006), Smith (2006) and Teklu et al. (2007).

LINSIG (2010) software generates signal timings for given flows; this software is often used in real life for junctions and small networks, and often involves relatively small scale routeing considerations.

1.3.5 Non-equilibrium route choice and signal control modelling

Hunt et al. (1982) developed the real time control system SCOOT; essentially from the TRANSYT model. Hu and Mahmassami (1997) studied, within a model, day to day evolution of network flows under real-time information and reactive signal control. Heydecker et al. (2004, 2007) propose an adaptive dynamic control system for traffic signals and also considered possible future objectives for traffic signal control. Smith and Mounce (2011) present a splitting rate model embracing in a simplified way both traffic re-routeing and signal control adjustments.

2. Responsive control in a simple network; following the equisaturation policy

The network in Figure 2 is signal-controlled and has a finite capacity.

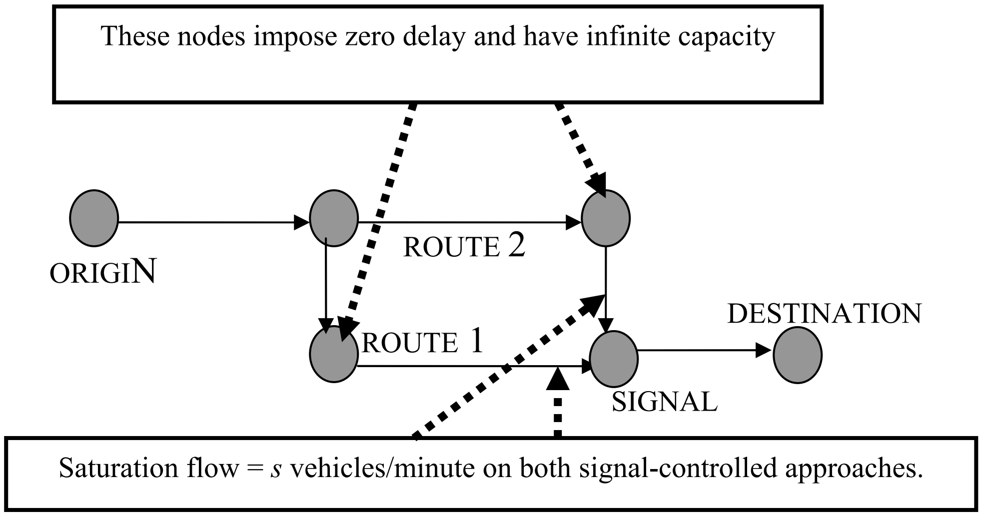

Figure 2. A simple symmetrical signal-controlled network. The network may be part of a large grid network. The signal follows the equisaturation policy.

We suppose that the equisaturation policy is used to set the signal in Figure 2. The signals are supposed here to adjust quickly and drivers are supposed to switch toward cheaper routes.

In this network in Figure 2 the two routes have the same undelayed travel time and the same saturation flow of s vehicles per minute at the signal. We suppose that there is a fixed (or rigid or steady) demand T (vehicles per minute) for travel from the origin to the destination. But we consider various feasible values of this demand T. It is easy to see that since the two approaches cannot be given green simultaneously, $T \leq s$. [We are here considering a steady state non-equilibrium model; and so all flow patterns considered are feasible steady state patterns.]

Let H_1 be the proportion of the fixed demand T choosing the lower route 1, let H_2 be the proportion of the fixed demand T choosing the upper route 2 and consider Figure 3.

A reasonable delay formula (specified in section 3 below) is used to obtain Figure 3. Here, with the equisaturation policy there are two stable equilibria as shown: all flow on route 1 or 2. In addition there are symmetrical equilibria; with equal flows along the two routes. These equilibria are unstable under natural assumptions (with the equisaturation policy). It follows from Figure 3 that the way any fixed OD demand spreads out over the links of this network will be subject to great uncertainty; such uncertainty is undesirable within a planning model. The model here is small; however the network may be part of a large network, and a large network may have many copies of networks similar to that shown in Figure 2.

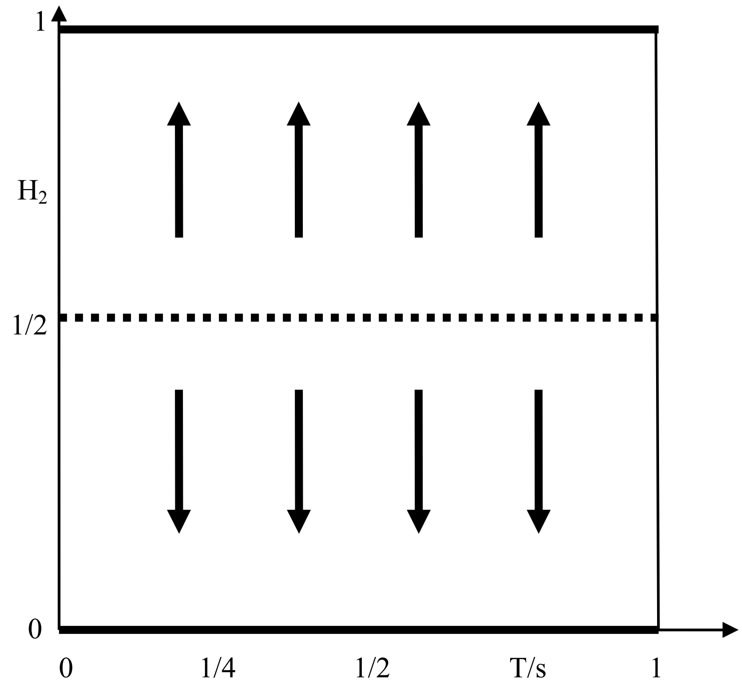

Figure 3. The set of (T/s, H2) pairs which are at equilibrium when the equisaturation policy controls the junction is as shown here in bold lines: dotted = unstable equilibria and solid = stable equilibria. The arrows show, for all feasible T, the natural direction of motion of non-equilibria as time passes.

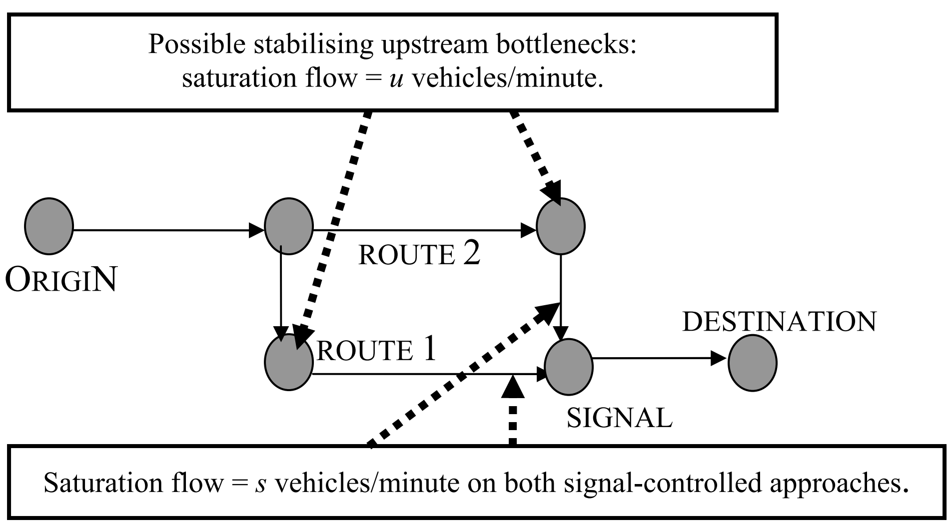

Figure 4. The same symmetrical network; but now with two upstream bottlenecks added.

3. Equisaturation control in the simple network with two upstream bottlenecks

It is clear from Figure 3 that network performance is very unpredictable: all traffic will at equilibrium be using either route 1 or route 2; the central symmetrical equilibria are all unstable.

One way of seeking to make the network more predictable may be to add two upstream bottlenecks as shown in Figure 4. This section explores this possibility.

Again the equisaturation policy is used to set the signals quickly and again the network flows are supposed to seek cheaper routes. But now two identical bottlenecks are present. These have equal capacities of u vehicles per minute. Can we choose u intelligently? Can we choose u to improve the stability of the central unstable equilibria and make network behaviour more predictable?

Let:

K_1 = uncongested or free-flow travel time along route 1 (ignoring bottleneck delays);

K_2 = uncongested or free-flow travel time along route 2 (ignoring bottleneck delays);

T = total steady OD flow via the routes in vehicles per minute (for feasibility $0 < T < \max\{s, 2u\}$);

H_1 = proportion of the Origin-Destination flow which travels along route 1;

H_2 = proportion of the Origin-Destination flow which travels along route 2;

G_1 = green time proportion awarded to route 1;

G_2 = green time proportion awarded to route 2;

d_{12} = delay felt at the signal by vehicles traversing route 1 (minutes per vehicle);

d_{22} = delay felt at the signal by vehicles traversing route 2 (minutes pr vehicle);

u = saturation flow at each "upstream" bottleneck (vehicles per minute);

d_{11} = "upstream" bottleneck delay on route 1 (minutes per vehicle);

d_{21} = "upstream" bottleneck delay on route 2 (minutes per vehicle);

C_1 = travel time or cost via route 1 = $K + d_{11} + d_{12}$ (minutes per vehicle);

C_2 = travel time or cost via route 2 = $K + d_{21} + d_{22}$ (minutes per vehicle);

We will suppose that

$$d_{12}=BTH_1/[sG_1(sG_1-TH_1)];$$

$$d_{22}=BTH_2/[sG_2(sG_2-TH_2)];$$

$d_{11}=BTH_1/[u(u–TH_1)]$ and

$d_{21}=BTH_2/[u(u–TH_2)]$.

Webster's famous two term delay formula d_{12W} is, in this context, as follows:

$$d_{12W} = 9/20\{cs(1- G_1)^2/(s - TH_1) + TH_1/[sG_1(sG_1 – TH_1)]\}$$

where c minutes is the cycle time of the signal. It can thus be seen that our chosen delay formula above is exactly the second term of Webster's delay formula when B = 9/20. With B = 9/20, the delay formula chosen in this paper will be very close to Webster's two-term delay formula when flows are close to capacity; since it is only the second term which is unbounded as $(sG_1 – TH_1) \rightarrow 0$. The first term here estimates the delay due to the stop-start nature of traffic signal operation (assuming that flow is steady). The second term allows for the random nature of arrivals.

[The Pollaczek-Khintchine (P-K) formula for the average waiting time W felt by a Poisson stream of arrivals (with arrival rate r) at a single server (with a constant service rate s) is as follows:

$$W = ½ r/[s(s – r)].$$

Thus the second term of Webster's formula is identical to the P-K formula if 9/20 is replaced by ½. See Pollaczek (1930), Khintchine (1932), Cohen (1969), Takacs (1971), Kingman (2009) and the Wikipedia entry for "Pollaczek-Khinchine formula" (consulted on 23.11.11).]

In this section, we are supposing that the signal moves the green time vector G quickly so as to equalise saturation ratios, so here, for all route-split vectors H, the green-time vector G satisfies:

$$H_1T/sG_1 = H_2T/sG_2.$$

It follows immediately that, for all feasible T:

$$G_1 = H_1 \text{ and } G_2 = H_2.$$

With these responsive green times G_1 and G_2:

$$d_{12} = BT/[s(sH_1 – TH_1)]$$

and

$$d_{22} = BT/[s(sH_2 – TH_2)].$$

In this initial example we suppose that $K_1 = K_1 = K$. Suppose now that flow seeks lower cost routes as day succeeds day. Then the direction of motion of the route split vector H will (at non-equilibria) be determined by the sign of C_1 - C_2. Now (remembering that $K_1 = K_1 = K$)

$$C_1 - C_2 = BT\{1/[s(sH_1–TH_1)] + H_1/[u(u–TH_1)] – 1/[s(sH_2–TH_2)] – H_2/[u(u–TH_2)]\}.$$

So:

$$
\begin{aligned}
&[s(sH_1-TH_1)][u(u-TH_1)][s(sH_2-TH_2)][u(u-TH_2)]\,[C_1 - C_2] \\
&= \;BT\,\{[u(u-TH_1)][s(sH_2-TH_2)][u(u-TH_2)] \\
&\qquad + H_1[s(sH_1-TH_1)][s(sH_2-TH_2)][u(u-TH_2)] \\
&\qquad - [s(sH_1-TH_1)][u(u-TH_1)][u(u-TH_2)] \\
&\qquad - H_2[s(sH_1-TH_1)][u(u-TH_1)][s(sH_2-TH_2)]\}. \qquad (1)
\end{aligned}
$$

Here we must suppose that T satisfies: $0 < T < \min\{2u, s\}$, so as to ensure that T is positive and feasible, or within the capacity of the network. Then $BT > 0$ and if further the route proportion vector H is feasible for this T then the multiplier of $C_1 - C_2$ on the left hand side of (1),

$$[s(sH_1-TH_1)][u(u-TH_1)][s(sH_2-TH_2)][u(u-TH_2)],$$

is positive because each of the four components in this product is then positive.

It now follows, by (1), that at each feasible (T, H), $C_1 - C_2 = C_1(T, H) - C_2(T, H)$ has the same sign as

$$
\begin{aligned}
&\{[u(u-TH_1)][s(sH_2-TH_2)][u(u-TH_2)] \\
&+ H_1[s(sH_1-TH_1)][s(sH_2-TH_2)][u(u-TH_2)] \\
&- [s(sH_1-TH_1)][u(u-TH_1)][u(u-TH_2)] \\
&- H_2[s(sH_1-TH_1)][u(u-TH_1)][s(sH_2-TH_2)]\}. \qquad (2)
\end{aligned}
$$

Expanding this expression (by multiplying out each of the four terms) and then leaving out all positive factors, the expression (2) above and so $C_1 - C_2$ has the same sign as

$$[H_1H_2(T^2 + sT - s^2) + (u^2 - uT)][H_2-H_1]. \qquad (3)$$

The regions where $C_1 - C_2 > 0$ and $C_1 - C_2 < 0$ may be separated by sets of equilibria where $C_1 - C_2 = 0$. It is clear from the factor $[H_2-H_1]$ in (3) above that one way of such equilibria arising is if

$$H_1 = H_2 = ½.$$

The other factor in (3) shows that another way of equilibria arising is if

$$A(H, T) = [H_1H_2(T^2 + sT - s^2) + (u^2 - uT)] = 0. \qquad (4)$$

So each equilibrium falls into at least one of the following three sets:

(1) the set of symmetrical equilibrium vectors H,
(2) equilibrium vectors H determined by equation (4), and
(3) equilibrium vectors H with all flow on a single least cost route.

To discuss non-equilibrium dynamics let us now suppose that

$$H_2 > H_1 > 0. \qquad (5)$$

This of course ensures that $H_1 \neq H_2$. In this case, we may divide (3) by $[H_2-H_1] > 0$. Thus if

$$A(H, T) = [H_1H_2(T^2 + sT - s^2) + (u^2 - uT)] > 0$$

then $C_1 - C_2 > 0$. Now with our assumption (5) there is positive flow on the more costly route 1 and flow will (under our hypothesis) swap from the higher cost route 1 to the lower cost route 2. Thus H_2 will *increase* (and H_1 will *decrease*).

The behaviour of the flows on the network will depend on u. For example, Figures 5 and 6 below show bifurcation diagrams corresponding to u=s/4 and u=s/2. In both figures, equilibria are shown as heavy lines and non-equilibrium directions of motion are indicated by heavy arrows. Figure 6 suggests that u = ½ may be an intelligent choice for u for this network.

Network performance may vary suddenly as the OD flow T changes. Suppose for example that for small T the flow pattern has $H_2 = 0$. Suppose now that T increases through s/4; the stable equilibrium with $H_2 = 0$ disappears and the central equilibrium with $H_2 = ½$ becomes attracting for all feasible H which are feasible (for each T > s/4). This value of u stabilises all symmetrical equilibria for 0 < T/s < ½; however this u does reduce network capacity to 2u = s/2 and substantially reduces the set of feasible (T/s, H_2).

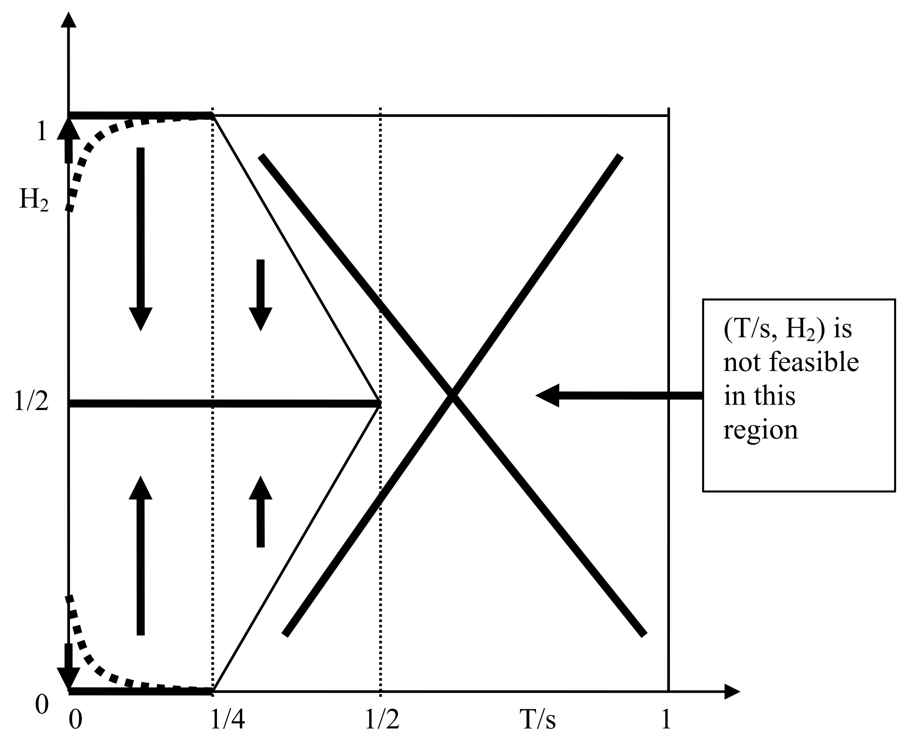

Figure 5. The set of (T/s, H_2) pairs which are at equilibrium when the equisaturation policy controls the junction and there are two upstream bottlenecks of capacity u = s/4 is shown here in bold lines: dotted = unstable equilibria and solid = stable equilibria. The arrows show the natural direction of motion of non-equilibria.

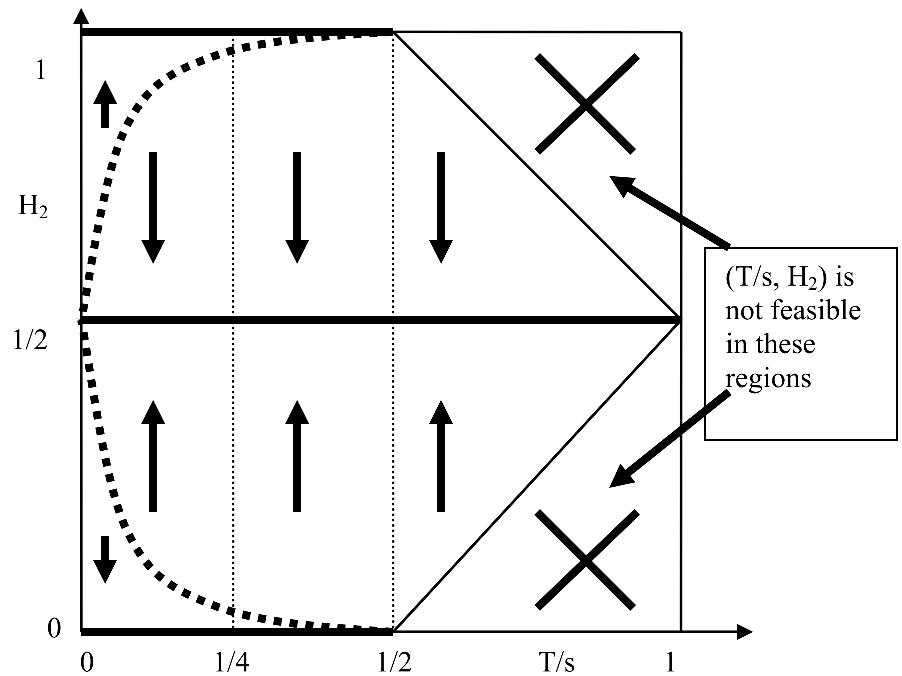

Figure 6. (T/s, H_2) diagram for u = s/2; again showing that network performance may vary significantly with load variations. This u does not reduce network capacity and stabilizes symmetrical equilibria for all T satisfying 0 < T < s.

By looking at Figures 5 and 6 it seems that u = s/2 is able to stabilise this network for all T such that 0 < T < s. There are still some uncertainties on the left of the figure, for small values of T, and as the OD load T varies. There also some restrictions on the route-split vector H for T/s > ½.

4. Responsive control in a simple asymmetrical network; following a modified P_0 policy

Now consider the similar, but asymmetrical, network in Figure 7 below (redrawn for clarity); this does not have additional bottlenecks. Here we suppose flows are controlled by using a control policy very similar to the P_0 policy (see Smith 1980). We call this policy P_1. Here there are bottlenecks only at the signal. No stabilisation will be needed or considered.

We show now that with the P_1 control policy the network shown in Figure 7 becomes stable under certain conditions. We assume that the saturation flow on the lower route at the signal is s_1 (vehicles per minute) where $s_1 < s_2$ (the saturation flow on the upper route). We also assume that route 1 is shorter than route 2. These assumptions are made for purely technical reasons: the stability arguments here seems likely to generalise to a wide class of networks with the same delay formulae.

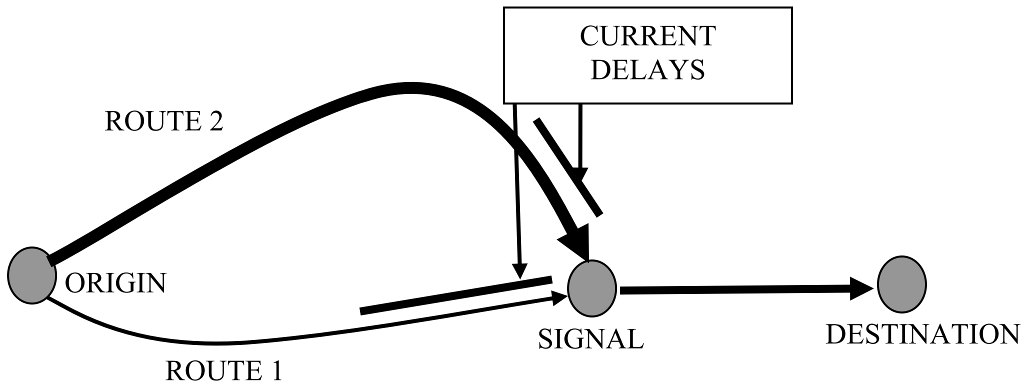

Figure 7. A simple asymmetrical signal-controlled network. Route 2 is longer and (ignoring delays) more costly than route 1. Route 2 is wider than route 1 at the signal. Here the signal follows the new P_1 policy. The lengths of the bars represent current delays. Uncongested travel costs via route 1 and 2 are K_1 minutes and K_2 minutes where $K_1 < K_2$.

We suppose as before that the delays d_1 on the lower route and d_2 on the upper route at the signal are derived as before from Webster's formula and are as follows:

$$d_1 = BTH_1/[s_1G_1(s_1G_1–TH_1)] = B/(s_1G_1–TH_1) – B/s_1G_1; \quad (6)$$

and

$$d_2 = BTH_2/[s_2G_2(sG_2–TH_2)] = B/(s_2G_2–TH_2) – B/s_2G_2. \quad (7)$$

This is just as in the previous symmetrical network at the junction.

Replace each green time proportion by 1 - (the corresponding red time proportion), or G_1 by $1 - R_1$ and G_2 by $1 – R_2$ and also let

$$f_1(x) = B/[s_1- x] \text{ and } f_2(x) = B/[s_2 – x]$$

for all real numbers $x \geq 0$. It follows that:

$$\begin{aligned} &f_1(s_1R_1 + TH_1) = B/[s_1 - (s_1R_1 + TH_1)] = B/(s_1G_1-TH_1),\\ &f_2(s_2R_2 + TH_2) = B/[s_2 – (s_2R_2 + TH_2)] = B/(s_2G_2-TH_2),\\ &f_1(s_1R_1) = B/[s_1 – s_1R_1] = B/s_1G_1 \text{ and}\\ &f_2(s_2R_2) = B/[s_2 - s_2R_2] = B/s_2G_2. \end{aligned} \quad (8)$$

It further follows from (6) and (7) that

$$\begin{aligned} &d_1 = f_1(s_1R_1 + TH_1) - f_1(s_1R_1) \text{ and}\\ &d_2 = f_2(s_2R_2 + TH_2) - f_2(s_2R_2). \end{aligned} \quad (9)$$

4.1 Equilibrium consistent with a modified P_0 policy

Here we will utilise a modification of the P_0 policy. Under the current circumstances, the usual formulation of the P_0 policy is: for any flows on route 1 and route 2, choose green-time proportions (or red time proportions) so that

$$s_1 d_1 = s_2 d_2.$$

The modification will apply this rule to only $f_1(s_1R_1 + TH_1)$ and $f_2(s_2R_2 + TH_2)$; these are only parts of the delay formulae d_1 and d_2 in (9) above. This modified version of P_0 is thus to be: for any flows H_1 and H_2 choose red time proportions$_1$ R_1 and R_2 so that

$$s_1 f_1(s_1R_1 + TH_1) = s_2 f_2(s_2R_2 + TH_2).$$

$s_1 f_1(s_1R_1 + TH_1)$ will sometimes be written s_1f_1 and $s_2f_2(s_2R_2 + TH_2)$ will sometimes be` written s_2f_2.

We here suppose that s_1, s_2, K_1, K_2 and T are such that there is a consistent equilibrium at which flows and red time proportions are all positive. In this case at this consistent equilibrium:

$$C_1 = C_2 \text{ and } s_1f_1 = s_2f_2. \tag{10}$$

4.2 Dynamics

Let $0 < T < s$. The set D of demand-feasible quadruples is defined below.

$$D = \{[H_1, H_2, R_1, R_2];\ H_1 + H_2 = 1,\ R_1 + R_2 = 1\}.$$

Also the set S of supply-feasible quadruples is as shown below.

$$S = \{[H_1, H_2, R_1, R_2];\ TH_1 + s_1R_1 < s_1, TH_2 + s_2R_2 < s_2,\ H_1 \geq 0,\ H_2 \geq 0,\ R_1 \geq 0,\ R_2 \geq 0\}.$$

Finally we suppose, largely for simplicity, that the ensuing trajectory generated below always lies in the interior of S.

There are many dynamical systems which would be natural to describe the non-equilibrium evolution of the system here. To be specific we suppose, most simply, that the starting (H, R) starts in D and in the interior of S and that (for $t > 0$):

$$\begin{aligned} dH_1/dt &= T(C_2 - C_1) = T\{K_2 + d_2 - [K_1 + d_1]\} \\ dH_2/dt &= T(C_1 - C_2) = T\{K_1 + d_1 - [K_2 + d_2]\} \\ dR_1/dt &= s_2f_2 - s_1f_1 \\ dR_2/dt &= s_1f_1 - s_2f_2. \end{aligned} \tag{11}$$

Let - F(H, R) stand for the right hand side of (11); then the whole dynamical system (11) may be written:

$$d[H, R]/dt = - F(H, R).$$

Dynamical system (11) is consistent with the two equilibrium Equations (10); inasmuch as all the variables H_1, H_2, R_1, R_2 do not vary under the dynamical system

(11) if and only if (10) holds. Thus equilibria of the dynamical system (11) are consistent equilibria as already defined above in (10).

The dynamical system (11) comprises only swaps of route-flow proportions and red time proportions; so the trajectory followed by (H, R)(t) as dynamical system (11) unfolds remains in D. *(We are supposing that it remains in the interior of S too.)*

We now outline a proof that, under these assumptions, any solution trajectory of the dynamical system (11) converges to a unique consistent equilibrium (H_1^*, H_2^*, R_1^*, R_2^*) in D. The key is to show that

F[H, R)] = - [the right hand side of equations (11)]

is a strictly monotone function of [H, R] = [H_1, H_2, R_1, R_2]. Some details of this proof are given in the appendix. Strict monotonicity implies that there is at most one equilibrium in $D \cap S$.

As shown in the appendix, strict monotonicity of F also ensures that the "kinetic energy"

$$V = ½\{[TC_1 – TC_2]^2 + [s_1f_1 – s_2f_2]^2\}$$

decreases (as (11) is followed) at each [H, R] where V[H, R] > 0. Assuming that the trajectory does not approach the boundary of S (by a variable becoming close to zero or by the capacity constraints in S becoming nearly violated) V will decline along the whole solution trajectory of (11). Lyapunov's theorem now applies: as time increases the solution (H, R)(t) converges to the single consistent equilibrium (H*, R*) as this is the only point at which the kinetic energy V is zero. (This is the unique minimum of V in $D \cap S$.)

The set-up and the arguments given here and in the appendix appear to be generalisable. After generalisation, they should apply to a general network with a more general version of dynamical system (11). However this paper does not state this more general dynamical system and does not give the general proof of stability. A related general stability result (with very different details and a very different setup) is given in Smith and Mounce (2011).

5. Conclusion

The paper shows, by considering a simple example network, that standard responsive controls may give rise to unpredictable behaviour in transport network models (and also in transport networks themselves); in our example there are two widely separated stable equilibria. The paper shows that for just the simple network here controlled by the equisaturation policy, this unpredictability may be ameliorated to a high degree by adding upstream bottlenecks.

The paper shows further that the responsive P_1 control on a similar network may be expected to behave in a much more predictable manner. Further the paper shows that natural dynamics involving both green-times and traffic flows, will in this case (under natural conditions) be stable. Further work is needed to extend the stability arguments here so that they apply to a general network with similar delay formulae, however these arguments, as presented here, do appear to be generalisable.

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7. Appendix: Proof that V declines to zero as (H, R) follows the dyamical system (11).

7.1. Assumptions

We assume the following conditions hold.

1. f_1 is a positive differentaible, increasing, convex real-valued function of a real variable.
2. f_2 is a positive differentaible, increasing, convex real-valued function of a real variable.
3. As [H, R](t) follows dynamical system (11), [H, R](t) never approaches the boundary of S.

7.2. Proof that V declines at non-equilibria (H, R)

The key to showing that (away from equilibrium) V declines along a solution trajectory of (11) is to show that

$$\begin{aligned} F[H, R] &= - [\text{the right hand side of equations (11)}] \\ &= - [T(C_2 - C_1), T(C_1 - C_2), s_2f_2 - s_1f_1, s_1f_1 - s_2f_2]^T \\ &= [T(C_1 - C_2), T(C_2 - C_1), s_1f_1 - s_2f_2, s_2f_2 - s_1f_1]^T \end{aligned}$$

is a *strictly monotone* function of $[H, R] = [H_1, H_2, R_1, R_2]$.

For any [H,R] in D and in the interior of S, F[H, R] is the projection onto the set D of the 4-vector

$$[TC_1, TC_2, s_1f_1, s_2f_2]^T = [T(K_1 + d_1), T(K_2 + d_2), s_1f_1, s_2f_2]^T. \qquad \text{(A1)}$$

Thus F is strictly monotone if (A1) is strictly monotone so consider (A1). Split this 4-vector (A1) into two parts:

$$[T(K_1 + d_1), s_1f_1] \text{ and } [T(K_2 + d_2), s_2f_2].$$

The first of these depends only on $[H_1, R_1]$ and the second depends only on $[H_2, R_2]$. To show that these are both strictly monotone we need only consider $T(K_1 + d_1), s_1f_1]$; a similar argument will then apply to $[T(K_2 + d_2), s_2f_2]$.

The vector

$$[T(K_1 + d_1), s_1f_1] = [TK_1 + Td_1, s_1f_1] = [TK_1, 0] + [Td_1, s_1f_1].$$

This is strictly monotone if and only if

$$F_1(H_1, R_1) = [Td_1, s_1f_1] = [T\{f_1(s_1R_1 + TH_1) - f_1(s_1R_1)\}, s_1f_1(s_1R_1 + TH_1)]$$

is strictly monotone.

Let $h > 0$ and consider moving from the point $[H_1, R_1]$ in the arbitrary direction $[\delta H_1, \delta R_1]$ to the point $[H_1, R_1] + h[\delta H_1, \delta R_1]$. The consequent change in $F_1(H_1, R_1)$ which is caused by the above change in $[H_1, R_1]$ is

$$\begin{aligned}&\Delta F_1(H_1, R_1)\\ &= [T\{f_1(s_1(R_1 + h\delta R_1) + T(H_1 + h\delta H_1)) - f_1(s_1(R_1 + h\delta R_1))\}, s_1f_1(s_1(R_1 + h\delta R_1) +\\ &T(H_1 + h\delta H_1))] - [T\{f_1(s_1R_1 + TH_1) - f_1(s_1R_1)\}, s_1f_1(s_1R_1 + TH_1)].\end{aligned}$$

The direction of motion $DF_1([H_1, R_1]; [\delta H_1, \delta R_1])$ of $F_1(H_1, R_1)$ as (H_1, R_1) moves in the direction $[\delta H_1, \delta R_1]$ is obtained by letting $h \rightarrow 0+$ in $\Delta F_1(H_1, R_1)/h$. We obtain:

$$\begin{aligned}&DF_1([H_1, R_1]; [\delta H_1, \delta R_1])\\ &= [T\{(s_1\delta R_1 + T\delta H_1)f_1'(s_1R_1 + TH_1) - s_1\delta R_1f_1'(s_1R_1)\}, s_1(s_i\delta R_1 + T\delta H_1)f_1'(s_1R_1 +\\ &TH_1)].\end{aligned}$$

$DF_1([H_1, R_1]; [\delta H_1, \delta R_1])$, the directional derivative of F_1 in the direction $[\delta H_1, \delta R_1]$ will be important in what follows.

The dot product of this directional derivative, $DF_1([H_1, R_1]; [\delta H_1, \delta R_1])$, with the direction $[\delta H_1, \delta R_1]$ is then given as follows:

$$\begin{aligned}&DF_1([H_1, R_1]; [\delta H_1, \delta R_1]) \cdot [\delta H_1, \delta R_1]\\ &=[T\{(s_1\delta R_1 + T\delta H_1)f_1'(s_1R_1 + TH_1) - s_1\delta R_1f_1'(s_1R_1)\}, s_1(s_1\delta R_1 + T\delta H_1)f_1'(s_1R_1 +\\ &TH_1)]\cdot [\delta H_1, \delta R_1]\end{aligned}$$

Let $a = f_1'(s_1R_1)$ and $b = f_1'(s_1R_1 + TH_1)$. Then

$$\begin{aligned}&DF_1([H_1, R_1]; [\delta H_1, \delta R_1]) \cdot [\delta H_1, \delta R_1]\\ &= \{bT^2(\delta H_1)^2 + (\delta H_1)(\delta R_1)[bTs_1 - aTs_1 + bTs_1] + bs_1^2(\delta R_1)^2\}\\ &= bT^2(\delta H_1)^2 + [2b - a]Ts_1(\delta H_1)(\delta R_1) + bs_1^2(\delta R_1)^2.\\ &\geq (b - a/2)T^2(\delta H_1)^2 + [2b - a]Ts_1(\delta H_1)(\delta R_1) + (b-a/2)s_1^2(\delta R_1)^2\\ &= (b - a/2)[T(\delta H_1) + s_1(\delta R_1)]^2 \qquad \text{(A2)}\\ &> 0\end{aligned}$$

if $(\delta H_1, \delta R_1) \neq 0$ and $b > a/2$, since (A2) is then a sum of squares of numbers some of which are positive and all are non-negative.

Now suppose that f_1 is non-negative, convex, and strictly increasing. Then f_1' is a non-decreasing positive function. It then follows that, always,

$$0 < a = f_1'(s_1R_1) \leq f_1'(s_1R_1 + TH_1) = b$$

and so always indeed

$$b > a/2$$

and it then follows by the inequality string containing (A2) that

$$DF_1([H_1, R_1]; [\delta H_1, \delta R_1]) \cdot [\delta H_1, \delta R_1] > 0.$$

This shows that $F_1[H_1, R_1]$ is a strictly monotone function of $[H_1, R_1]$; and we will write this as follows. For all $[H_1, R_1]$ and non-zero $[\delta H_1, \delta R_1]$:

$$DF_1([H_1, R_1]; [\delta H_1, \delta R_1]) \cdot [\delta H_1, \delta R_1] > 0.$$

Strict monotonicity is inherited by Cartesian products and so

$$[T(K_1 + d_1), T(K_2 + d_2), s_1 f_1, s_2 f_2]$$

is strictly monotone. Strict monotonicity is also inherited by projections and so, further,

$$F(H, R) = \mathrm{Proj}_D[T(K_1 + d_1), T(K_2 + d_2), s_1 f_1, s_2 f_2].$$

is strictly monotone.

Now let the kinetic energy of the dynamical system (11) be V where:

$$V = [TC_1 - TC_2]^2 + [s_1 f_1 - s_2 f_2]^2 = ½ [-F(H, R)]^T[-F(H, R)] = ½ [F(H, R)]^T[F(H, R)].$$

We show that V declines in the direction –F.

Strict monotonicity of F implies that the Jacobian matrix J of F is positive definite everywhere. We will use this to show that V declines along a solution of (11), away from equilibrium.

Let

$$x = (H_1, H_2, R_1, R_2)$$

be a non-equilibrium. Then $J = J(x)$ is positive definite, (11) may be written:

$$dx/dt = - F(x)$$

(where F(x) is a non-zero vector) and also

$$V(x(t)) = V(x) = ½ F(x)^T F(x) > 0.$$

Using this notation it now follows that

$$dV/dt = \mathrm{grad}V(x) \cdot dx/dt = [J^T(x)F(x)] \cdot (- F(x)) = F(x)^T J(x) (- F(x)) = - F(x)^T J F(x) < 0$$

since J is positive definite and $F(x) \neq 0$.

7.3. V declines to zero and x(t) converges to equilibrium

Provided x(t) does not approach the boundary of S, $dV/dt < 0$ as x(t) moves along the dynamical system (11). In this case, by letting $t \rightarrow \infty$, the standard Lyapunov argument shows that:

(1) $V(x(t)) \rightarrow 0$,
(2) the set E of points x in $D \cap S$ with $V(x) = 0$ is non-empty and
(3) $\mathrm{dist}(x(t), E) \rightarrow 0$.

By strict monotonicity of F, the set E of equilibria contains just one point $x^* = [H_1^*, H_2^*, R_1^*, R_2^*]$ and so $x(t) \rightarrow x^*$ as $t \rightarrow \infty$. Convergence to equilibrium has been proved.

8. References

Aashtiani, H. and T., Magnanti, 1983. Equilibria on a congested transportation network. *SIAM Journal on Algebraic and Discrete Methods*, 2(2), 213-226.

Allsop, R. E., 1974. Some possibilities for using Traffic Control to Influence Trip Distribution and Route Choice, in (Ed.) Buckley, D. J., *Proceedings of the Sixth International Symposium on Transportation and Traffic Theory*, Sydney, Elsevier, 345-374.

Allsop R. E. and J.A., Charlesworth, 1977. Traffic in a signal controlled road network: an example of different signal timings inducing different routeings. *Traffic Engineering and Control*, 18(5), 262-264.

Bar-Gera, H., 2002. Origin-based algorithm for the traffic assignment problem. *Tansportation Science*, 36(4), 398 – 417.

Bar-Gera, H. and D., Boyce, 2003. Origin-based algorithms for combined travel forecasting models. *Transportation Research Part B*, 37(5), 405-422.

Bar-Gera, H. and D., Boyce, 2006. Solving a non-convex combined travel forecasting model by the method of successive averages with constant step sizes. *Transportation Research Part B*, 40(5), 351-367.

Bar-Gera, H., 2010. Traffic assignment by paired alternative segments. *Transportation Research Part B*, 44(8-9), 1022-1046.

Beckmann, M., C. B., McGuire and C. B., Winsten, 1956. Studies in the Economics of *Transportation*. Yale University Press, New Haven, CT.

Bell, M. G. H., 1992. Future directions in traffic signal control. *Transportation Research Part A*, 26(4), 303-313.

Bellei, G., G., Gentile, and N., Papola, 2005. A within-day dynamic traffic assignment model for urban road networks. *Transportation Research Part B*, 39(1), 1–29.

Bentley, R. W. and T. A., Lambe, 1980. Assignment of traffic to a network of signalized city streets. *Transportation Research Part A*, 14(1), 57-65.

Bie, J. and H. K., Lo, 2010. Stability and attraction domains of traffic equilibria in a day-to-day dynamical system formulation. *Transportation Research Part B*, 44(1), 90–107.

Cantarella, G. E., 1997. A general fixed-point approach to multimodal multi-user equilibrium assignment with elastic demand. *Transportation Science*, 31(2), 107-128.

Cantarella, G. and E., Cascetta, 1995. Dynamic processes and equilibrium in transportation networks: Towards a unifying theory. *Transportation Science*, 29(4), 305–329.

Carey, M., 1987. Optimal time-varying flows on congested networks. *Operations Research*, 35(1), 58–69.

Cascetta, E., 1991. A day-to-day and within-day dynamic stochastic assignment model. *Transportation Research Part A*, 25(5), 277 – 291.

Cascetta, E., 2009. *Transportation Systems Analysis.* Springer, New York.

Cascetta, E., M., Gallo, and D., Montella, 2006. Models and algorithms for the optimisation of sugnal settings on urben networks with stochastic assignment models. *Annals of Operations Research*, 144(1), 301–328.

Cipriano, E. and G., Fusco, 2004. Combined Signal Setting Design and Traffic Assignment Problem. *European Journal of Operations Research*, 155(3), 569–583.

Clegg, J., M. J., Smith, Y., Xiang and R., Yarrow, 2001. Bilevel programming applied to optimising urban transportation. *Transportation Research Part B*, 35(1), 41-70.

Cohen, J. W., 1969. The Single Server Queue. North Holland, Amsterdam and London.

Dafermos, S., 1980. Traffic equilibrium and variational inequalities. *Transportation Science*, 14(1), 42–54.

de Palma, A., M., Ben Akiva, C., Lefevre, and N., Litinas, 1983. Stochastic equilibrium model of peak period traffic congestion. *Transportation Science*, 17(4), 430 – 453.

Dickson, T. J., 1981. A note on traffic assignment and signal timings in a signal-controlled road network. *Transportation Research Part B*, 15(4), 267 – 271.

Fisk, C. S., 1980. Some developments in equilibrium traffic assignment. *Transportation Research Part B*, 14(3), 243-255.

Fisk, C. S., 1984. Game theory and transportation system modelling. *Transportation Research Part B*, 18(4), 301-313.

Friesz, T. L., C., Kwon, D., Bernstein, 2008. Analytical Dynamic Traffic Assignment Models. *Handbook of Transport Modelling*, in (Eds.) Hensher, D. A. and K. J., Button, Elsevier, 221–236.

Friesz, T. L. and R., Mookherjee, 2006. Solving the dynamic user equilibrium problem with state dependent time shifts. *Transportation Research Part B*, 40(3), 207-229.

Gartner, N. H., 1976. Area traffic control and network equilibrium. *Traffic Equilibrium methods,* in (Ed.) M. Florian, *Lecture Notes in Economics and Mathematical Statistic.118*, Springer-Verlag, Berlin, 274–297.

Gartner, N. H., 1983. OPAC: A demand responsive strategy for traffic signal control. *Transportation Research Record* 906, 75-81.

Gentile, G., 2009. On the accurate convergence of deterministic assignment when comparing scenarios for large networks; investigating the LUCE algorithm. Paper presented at the *SIDT 2009 International Conference*, Milan, June.

Gentile, G. and K., Noekel, 2009. Linear user cost equilibrium: the new algorithm for traffic assignment in VISUM. Paper presented at the *European Transport Forum,* Brussels, September.

Hamdar, S. H., M., Treiber, H. S., Mahmassami and A., Kesting, 2008. Modeling driver behaviour as a sequential risk-taking task. *Transportation Research Record*, 2066, 208-217.

He, X., X., Guo and H., Liu, 2010. A link-based day to day traffic assignment model. *Transportation Research Part B*, 44(4), 597 – 608.

Heydecker, B. G., C., Chen, and C. K., Wong, 2007. Adaptive dynamic control for road traffic signal. *IEEE International Conference on Networking, Sensing and Control*, London.

Heydecker, B. G., 2004. Objectives, stimulus and feedback in signal control of road traffic. *ITS Journal*, 8(2), 63-76

Hu, T-Y and H. S., Mahmassani, 1997. Day-to-day evolution of network flows under real-time information and reactive signal control. *Transportation Research Part C*, 5(1), 51-69.

Hunt, P. B., D. I., Robertson, R. D., Bretherton and M. C., Royle, 1982. The SCOOT on-line traffic optimisation technique. *Traffic Engineering and Control*, 23, 190–192.

Khintchine, A. Y., 1932. Mathematical theory of a stationary queue. *Mathematicheskii Sbornik*, 39(4) 73–84.

Kingman, J., F., C., 2009. The first Erlang century – and the next. *Queueing Systems*, 63, 3-4.

Kupiszewska, D. and D., Van Vliet, 1999. 101 uses for path-based assignment. *Proceedings of the European Transport Conference*, PTRC Education and Research Services Ltd. Seminar F, Transportation Planning Methods, 121-132.

LINSIG (2010). http://en.wikipedia.org/wiki/LINSIG (accessed on 14.08.10).

Lyapunov, A. M., 1907. Probleme general de la stabilite de mouvement, *Annuals of Faculty of Science. Toulouse*, 9, 203–474. Reprinted in *Annuals of Mathematic Studies*. No 17 (Princeton University Press), 1949.

Mahmassami, H. S. and Liu, Y.-H., 1999. Dynamics of commuting decision behaviour under advanced traveler information systems. *Transportation Research Part C*, 7(2-3), 91-107.

Marcotte, P., 1983. Network optimisation with continuous control parameters. *Transportation Science*, 17(2), 181-197

Meneguzzer, C., 1996. Computational experiments with a combined traffic assignment and control model with asymmetric cost functions, in (Eds). Stephanedes, Y. J. and F., Filippi, *Proceedings of the 4th International Conference on Applications of Advanced Technologies in Transportation Engineering*, ASCE, New York, 609-614.

Meneguzzer, C., 1997. Review of models combining traffic assignment and signal control. *ASCE Journal of Transportation Engineering*, 123(2), 148-155.

Mounce, R., 2009. Existence of equilibrium in a continuous dynamic queueing model for traffic networks with responsive signal control, in (Eds.) Lam, W. H. K., S. C., Wong and H. K., Lo, *Proceedings of the Eighteenth International Symposium on Transportation and Traffic Theory*, Hong Kong Polytechnic University, Hong Kong, Springer, 327-344,

Pollaczek, F., 1930. Uber eine Aufgabe der Wahrscheinlichkeitstheorie. *Mathematische Zeitschrift*, 32, 64–100.

Robertson, D. I., 1969. TRANSYT: a traffic network study tool. *RRL Lab. report LR253*, Road Research Laboratory, Crowthorne, UK.

Sheffi, Y., 1985. Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods, Prentice Hall (New Jersey) and available on-line.

Sheffi, Y. and W. B., Powell, 1983. Optimal Signal Setting over Transportation Networks, *Transportation Engineering*, 109(6), 824 – 839.

Smith, M. J., 1979a. A local traffic control policy which automatically maximises the overall travel capacity of an urban road network. *Proceedings of the International Symposium on Traffic Control Systems*, University of California, Berkeley, Volume 2A, 11-32; and in *Traffic Engineering and Control*, 21, 1980, 298-302.

Smith, M., J., 1979b. The existence, uniqueness and stability of traffic equilibria. *Transportation Research Part B*, 13(4), 295-304.

Smith, M. J., 1984a. The stability of a dynamic model of traffic assignment - an application of a method of Lyapunov. *Transportation Science*, 18(3), 245 - 252.

Smith, M. J., 1984b. A descent algorithm for solving a variety of monotone equilibrium problems, in (Eds.) Volmuller, J. and R., Hamerslag, *Proceedings of the Ninth International Symposium on Transportation and Traffic Theory*, Delft University of Technology, Netherlands, VNU Science Press, 273-297.

Smith, M. J., 1984c. A Descent Method for Solving Monotone Variational Inequalities and Monotone Complementarity Problems. *Journal of Optimization Theory and Applications*, 44, 485-496.

Smith, M. J., 1987. Traffic control and traffic assignment in a signal controlled network with queueing, in (Eds.) Gartner, N. H. and N. H. M., Wilson, *Proceedings of the Tenth International Symposium on Transportation and Traffic Theory*, MIT, Cambridge, Massachusetts, Elsevier, 61-68.

Smith, M. J. and T., van Vuren, 1993. Traffic equilibrium with responsive traffic control. *Transportation Science*, 27(2), 118-132.

Smith, M. J., 2006. Bilevel optimisation of prices and signals in transportation models, in (Eds.) Lawphongpanich, S., D. W., Hearn, and M. J., Smith, *Mathematical and Computational Models for Congestion Charging*, 159–199.

Smith, M. J., 2010. Intelligent Network Control: using an Assignment - Control Model to Design Fixed Time Signal Timings, in (Eds.) Tampere, C., F., Viti, and L., Immers, Edward Elgar, *Transport Planning – Advances in Dynamic Traffic Assignment*, 57–72.

Smith, M. J. and R., Mounce, 2011. A splitting rate model of traffic re-routeing and traffic control. *Transportation Research Part B*, 1389-1409. (Also in (Eds) Cassidy, M. and C. F., Daganzo, *Proceedings of the Nineteenth International Symposium on Transportation and Traffic Theory*, Berkeley, Elsevier, 316–340.

Taale, H. and H., J., van Zuylen, 2001. The combined traffic assignment and control problem – an overview of 25 years of research. *Proceedings of the 9th World Confernce on Transport Research* (WCTR), Seoul, Korea, 22-27.

Takacs, L., 1971. Review: J. W. Cohen, The Single Server Queue. *Annals of Mathematical Statistics*, 42(6), 2162 – 2164.

Teklu, F., A., Sumalee and D., Watling, 2007. A genetic algorithm approach for optimising traffic signals considering routeing. *Computer aided Civil and Infrastrucrture Engineering*, 22(1), 31-43.

Van Vuren, T. and D., Van Vliet, 1992. Route Choice and Signal Control. Avebury.

Wardrop, J. G., 1952. Some Theoretical Aspects of Road Traffic Research, *Proceedings, Institution of Civil Engineers II* (1), 235-278.

Watling, D., 1996. Asymmetric problems and stochastic process models of traffic assignment. *Transportation Research Part B*, 30(5), 339–357.

Watling, D., 1999. Stability of the stochastic assignment problem, a dynamical systems approach. *Transportation Research Part B*, 33(4), 281–312.

Watling, D. and M., Hazelton, 2003. The dynamics and equilibria of day-to-day assignment models. *Networks and Spatial Economics*, 3(3), 349–370.

Webster, F. V., 1958. *Traffic Signal Settings, Department of Transport*, Road Research Technical Paper No. 39, HMSO, London.

Wood, K., 1993. Urban Traffic Control, Systems Review. Project Report 41, *Transport Research Laboratory*, Crowthorne, United Kingdom.

Yang, H. and S., Yagar, 1995. Traffic assignment and signal control in saturated road networks. *Transportation Research Part A*, 29(2), 125-139.

Yang, H., 1996. Equilibrium network traffic signal setting under conditions of queueing and congestion in (Eds.) Stephanedes, Y. J. and F., Filippi, *Applications of Advanced Technologies in Transportation Engineering*, American Society of Civil Engineers, 578-582.